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Multiple Refinable Hermite Interpolants[1]

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We consider solutions of a system of refinement equations written in the form as

$$\phi(x) = \sum_{n \in \mathbb{Z}} a(n)\, \phi(2x-n),$$

where $\phi = (\phi_1, ..., \phi_r)^T$ is a vector of compactly supported functions on $\mathbb{R}$ and a is a finitely supported sequence of $r \times r$ matrices called the refinement mask. If ϕ is a continuous solution and a is supported on $[N_1, N_2]$, then $v := (\phi(n))_{n=N_1}^{N_2-1}$ is an eigenvector of the matrix $(a(2k-n))_{k,n=N_1}^{N_2-1}$ associated with eigenvalue 1. Conversely, given such an eigenvector v, we may ask whether there exists a continuous solution ϕ such that $\phi(n) = v(n)$ for $N_1 \leqslant n \leqslant N_2 - 1$ ($\phi(n) = 0$ for $n \notin [N_1, N_2 - 1]$, according to the support). The first part of this paper answers this question completely. This existence problem is more general than either the convergence of the subdivision scheme or the requirement of stability, since in one of the latter cases, the eigenvector v is unique up to a constant multiplication. The second part of this paper is concerned with Hermite interpolant solutions, i.e., for some $n_0 \in \mathbb{Z}$ and $j, m = 1, ..., r$, $\phi_j \in C^{r-1}(\mathbb{R})$ and $\phi_j^{(m-1)}(n) = \delta_{j,m} \delta_{n,n_0}$, $n \in \mathbb{Z}$. We provide a necessary and sufficient condition for the refinement equation to have an Hermite interpolant solution. The condition is strictly in terms of the refinement mask. Our method is to characterize the existence and the Hermite interpolant condition by joint spectral radii of matrices. Several concrete examples are presented to illustrate the general theory. © 2000 Academic Press

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1. INTRODUCTION

Vector subdivision schemes play an important role in the construction of multiple wavelets, and the design of curves and surfaces in CAGD. The

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limit ϕ of a convergent vector subdivision scheme satisfies a *refinement equation*

$$\phi(x) = \sum_{n \in \mathbb{Z}} a(n)\, \phi(2x - n), \tag{1.1}$$

where each $a(n)$ is an $r \times r$ matrix of complex numbers and $a(n) = 0$ except for finitely many n. We view a as a sequence from $\mathbb{Z}$ to $\mathbb{C}^{r \times r}$ and call it the *refinement mask*.

Let $\phi_1, ..., \phi_r$ be compactly supported distributions on $\mathbb{R}$. Denote by ϕ the vector $(\phi_1, ..., \phi_r)^T$, the transpose of $(\phi_1, ..., \phi_r)$. We say that ϕ is a *multiple refinable function* if it satisfies a refinement equation of the form (1.1).

In the scalar case ($r = 1$), a continuous solution ϕ of (1.1) is called a refinable interpolant, if $\phi(0) = 1$ and $\phi(n) = 0$ for all $n \in \mathbb{Z} \backslash \{0\}$. The study of refinable interpolants and interpolatory subdivision schemes was initiated by Deslauriers and Dubuc [6]. It turns out that the orthogonal and biorthogonal wavelets are closely related to refinable interpolants, see the work of Micchelli [22]. So the study of refinable interpolants plays an essential role in wavelet analysis.

In this paper we are interested in the vector case ($r \geqslant 1$). Our main purpose is to characterize compactly supported multiple refinable functions ϕ which satisfy the following condition for some $n_0 \in \mathbb{Z}$,

$$\phi_j \in C^{r-1}(\mathbb{R}) \qquad \text{and} \qquad \phi_j^{(m-1)}(n) = \delta_{j,m} \delta_{n,n_0}, \\ \forall j, m = 1, ..., r, \quad n \in \mathbb{Z}. \tag{1.2}$$

In this case, ϕ is called a *multiple refinable Hermite interpolant* at n_0.

Our characterization is based on the joint spectral radius which was defined by Rota and Strang [26] and was introduced into the investigation of wavelets by Daubechies and Lagarias [5].

Let V be a *finite-dimensional* vector space equipped with a vector norm $\|\cdot\|$. For a linear operator A on V, define

$$\|A\| := \max_{\|v\| = 1} \{\|Av\|\}.$$

Let $\mathscr{A}$ be a finite multiset of linear operators on V. Set

$$\|\mathscr{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\| : A_1, ..., A_n \in \mathscr{A}\}.$$

Then the *joint spectral radius* of $\mathscr{A}$ is defined to be

$$\rho_\infty(\mathscr{A}) := \lim_{n \to \infty} \|\mathscr{A}^n\|_\infty^{1/n}. \tag{1.3}$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathscr{A}^n\|_\infty^{1/n} = \inf_{n\geqslant 1} \|\mathscr{A}^n\|_\infty^{1/n}.$$

Clearly, $\rho_\infty(\mathscr{A})$ is independent of the choice of the vector norm on V.

If $\mathscr{A}$ consists of a single linear operator A, then $\rho_\infty(\mathscr{A}) = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A. It is easily seen that $\rho(A) \leqslant \rho_\infty(\mathscr{A})$ for any element A in $\mathscr{A}$.

The joint spectral radius is hard to compute if one uses the definition (1.3), since the limit in (1.3) is reached very slowly. An efficient way to do is to apply the p-norm joint spectral radius introduced by Jia [15]: first use the formula provided by Zhou [30] to compute p-norm joint spectral radius in terms of the spectral radius of some finite matrix when p is an even integer; then estimate the joint spectral radius by its relation to the p-norm joint spectral radius found by Strang and Zhou in [27].

Our main results involve the joint spectral radius of some linear operators restricted to certain common invariant subspaces, which often reduces the complexity of computation. Now let $\mathscr{A}$ be a finite multiset of linear operators on a normed vector space V, which is not necessarily finite dimensional. A subspace W of V is said to be invariant under $\mathscr{A}$, or *$\mathscr{A}$-invariant*, if it is invariant under every operator A in $\mathscr{A}$. For a vector $w \in V$, we define

$$\|\mathscr{A}^n w\|_\infty := \max\{\|A_1 \cdots A_n w\| : A_1, ..., A_n \in \mathscr{A}\}.$$

If the minimal $\mathscr{A}$-invariant subspace containing w, denoted as $V(w)$, is finite dimensional, then we have

$$\rho_\infty(\mathscr{A}|_{V(w)}) = \lim_{n\to\infty} \|\mathscr{A}^n w\|_\infty^{1/n} = \inf_{n\geqslant 1} \|\mathscr{A}^n|_{V(w)}\|_\infty^{1/n}. \tag{1.4}$$

See Han and Jia [12, Lemma 2.4] for a proof of this result. In fact, the proof of Lemma 2.4 in [12] shows that there exists a positive constant C such that for all $n \in \mathbb{N}$,

$$\|\mathscr{A}^n|_{V(w)}\|_\infty \leqslant C \|\mathscr{A}^n w\|_\infty. \tag{1.5}$$

Suppose that W is an $\mathscr{A}$-invariant subspace of V with $\dim W = m < \infty$. Let $w \in W$. Then $V(w) \subset W$ and $V(w)$ is spanned by

$$\{w\} \cup \{A_1 \cdots A_l w : A_1, ..., A_l \in \mathscr{A}, 1 \leqslant l \leqslant m-1\}.$$

In our case, the elements in $\mathscr{A}$ are two linear operators A_ε, $\varepsilon = 0, 1$, on $(\ell_0(\mathbb{Z}))^r$, the linear space of all finitely supported sequences of $r \times 1$ vectors, given by

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}} a(\varepsilon + 2\alpha - \beta)\, v(\beta), \qquad \alpha \in \mathbb{Z}, \quad v \in (\ell_0(\mathbb{Z}))^r. \tag{1.6}$$

Here and throughout the paper, we assume that the refinement mask a is in $(\ell_0(\mathbb{Z}))^{r \times r}$, the linear space of all finitely supported sequences of $r \times r$ complex matrices.

For a bounded subset K of $\mathbb{R}$, denote $\ell(K)$ as the linear space of all sequences supported in $K \cap \mathbb{Z}$. Suppose that a is supported in $[N_1, N_2]$ with $N_1 < N_2$, two integers. Then, for $j \leqslant N_1$ and $k \geqslant N_2 - 1$, $(\ell([j, k]))^r$ is invariant under both A_0 and A_1. In particular, $(\ell([N_1, k]))^r$ is an invariate subspace of A_0 for any $k \geqslant N_2 - 1$. Consequently, for $w \in (\ell([s, t]))^r$, the minimal common invariant subspace under A_0 and A_1 containing w is a subspace of $(\ell([N'_1, N'_2]))^r$ with $N'_1 := \min\{N_1, s\}$, $N'_2 := \max\{N_2 - 1, t\}$, and is generated by

$$\{w\} \cup \{A_{\varepsilon_1} \cdots A_{\varepsilon_l} w : \varepsilon_1, ..., \varepsilon_l \in \{0, 1\}, 1 \leqslant l \leqslant r(N'_2 - N'_1 + 1) - 1\}.$$

To characterize a multiple refinable Hermite interpolant at n_0, the *first step* is to find a compactly supported continuous solution ϕ of (1.1) with $\{\phi(n)\}_{n \in \mathbb{Z}}$ specified as the sequence $v \in (\ell_0(\mathbb{Z}))^r$ given by $v(n) = (1, 0, ..., 0)^T \delta_{n, n_0}$. Observe that if ϕ is a nonzero compactly supported continuous solution of (1.1), then $\{v(n) = \phi(n)\}_{n \in \mathbb{Z}} \in (\ell_0(\mathbb{Z}))^r$ is an eigenvector of A_0 associated with the eigenvalue 1:

$$v(n) = \phi(n) = \sum_{l \in \mathbb{Z}} a(l)\, \phi(2n - l) = \sum_{l \in \mathbb{Z}} a(2n - l)\, \phi(l) = A_0 v(n), \qquad \forall n \in \mathbb{Z}.$$

Thus, the first task for characterizing multiple refinable Hermite interpolants is carried out, once we solve the following problem: Given $a \in (\ell_0(\mathbb{Z}))^{r \times r}$ and an eigenvector $v \in (\ell_0(\mathbb{Z}))^r$ of the linear operator A_0 associated with the eigenvalue 1, when does there exist a vector ϕ of compactly supported continuous functions on $\mathbb{R}$ such that (1.1) holds and $\phi(n) = v(n)$ for all $n \in \mathbb{Z}$? In Section 2, we answer this question, and a necessary and sufficient condition is that $\rho_\infty(\mathscr{A}\,|_{V(\nabla v)}) < 1$, where $\rho_\infty(\mathscr{A}\,|_{V(\nabla v)})$ is the joint spectral radius of $\mathscr{A} = \{A_0, A_1\}$ restricted to the minimal common invariant subspace under A_0 and A_1 containing $v - v(\cdot - 1) \in (\ell_0(\mathbb{Z}))^r$.

After a compactly supported continuous solution ϕ of (1.1) has been found with $\phi(n) = (1, 0, ..., 0)^T \delta_{n, n_0}$ for all $n \in \mathbb{Z}$, the *next requirement* is that

ϕ is in $(C^{r-1}(\mathbb{R}))^r$, the linear space of all $r \times 1$ vectors of C^{r-1} functions on $\mathbb{R}$. In Section 3 we show how to check this requirement by finding the optimal smoothness of a compactly supported continuous solution ϕ of (1.1) with $\phi(n) = v(n)$ for all $n \in \mathbb{Z}$, where v is an eigenvector of A_0 associated with the eigenvalue 1. This optimal smoothness is measured by $-\log_2 \rho_\infty(\mathscr{A}|_{V(\nabla^k v)})$ (called the critical exponent), and does not require stability.

The *last condition* for a multiple refinable Hermite interpolant is that the function values and derivatives satisfy (1.2). In Section 3 we will reduce this condition into a requirement on a finite matrix which is the restriction of A_0 onto a finite interval. This in connection with our previous two discussions provides a complete characterization for a refinement equation to have a multiple refinable Hermite interpolant solution, see Theorem 5.

Finally, in Section 4, we present some examples of multiple refinable Hermite interpolants to illustrate the general theory.

2. EXISTENCE OF CONTINUOUS SOLUTIONS

In this section we give a characterization for the existence of compactly supported continuous solutions of the refinement equations. For the existence of compactly supported distribution solutions, see [13, 29].

Recall the definition (1.6) of A_0 and A_1. In what follows we always set $\mathscr{A} = \{A_0, A_1\}$.

Denote by ∇ the difference operator on $\ell_0(\mathbb{Z})$,

$$\nabla v := v - v(\cdot - 1), \qquad v \in \ell_0(\mathbb{Z}).$$

The domain of the difference operator ∇ can be naturally extended to include $(\ell_0(\mathbb{Z}))^r$ and $(\ell_0(\mathbb{Z}))^{r \times r}$.

If $\phi \in (C(\mathbb{R}))^r$ is a nonzero compactly supported solution of (1.1), then

$$\phi(n) = \sum_{l \in \mathbb{Z}} a(2n - l)\, \phi(l), \qquad n \in \mathbb{Z}.$$

That is, the sequence $v \in (\ell_0(\mathbb{Z}))^r$ given by $v(n) = \phi(n)$, $n \in \mathbb{Z}$, is an eigenvector of A_0 associated with eigenvalue 1.

Conversely, given an eigenvector $v \in (\ell_0(\mathbb{Z}))^r$ of A_0 assoiated with eigenvalue 1, the existence of a compactly supported continuous solution ϕ with $\phi|_{\mathbb{Z}} = v$ can be characterized as follows.

THEOREM 1. *Let $a \in (\ell_0(\mathbb{Z}))^{r \times r}$ and $v \in (\ell_0(\mathbb{Z}))^r$ be an eigenvector of A_0 associated with eigenvalue* 1. *Then there exists a vector $\phi = (\phi_1, ..., \phi_r)^T$ of* *compactly supported continuous functions on* $\mathbb{R}$ *such that* (1.1) *holds and* $\phi(n)=v(n)$ *for every* $n\in\mathbb{Z}$, *if and only if*

$$\rho_\infty(\mathscr{A}\,|_{V(\nabla v)})<1. \tag{2.1}$$

In this case, v *is supported in* $[N_1+1, N_2-1]$ *if* a *is supported in* $[N_1, N_2]$.

To prove this result, we use the sequence $\{a_n\}_{n\in\mathbb{N}}$ in $(\ell_0(\mathbb{Z}))^{r\times r}$ given by

$$a_1=a \qquad \text{and} \qquad a_{n+1}(\alpha)=\sum_{\beta\in\mathbb{Z}} a_n(\beta)\,a(\alpha-2\beta), \qquad \alpha\in\mathbb{Z}, \quad n\in\mathbb{N}. \tag{2.2}$$

This sequence was introduced by Jia, Riemenschneider and Zhou in [18]. It plays a crucial role in our investigation of convergence of vector subdivision schemes in [18] and smoothness of multiple refinable functions in [19]. In the scalar case ($r=1$), this sequence can be considered as iterations of subdivision operators investigated in detail by Cavaretta, Dahmen, and Micchelli [1]. Note that the order of matrix multiplications here is essential, which is quite helpful to our study.

Suppose that ϕ is a vector of compactly supported distributions on $\mathbb{R}$ satisfying (1.1), then by iterating (1.1) n times,

$$\phi(x)=\sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\,\phi(2^n x-\alpha), \qquad n\in\mathbb{N}. \tag{2.3}$$

For $w\in(\ell_0(\mathbb{Z}))^r$, it was shown in [18] that

$$\|\mathscr{A}^n w\|_\infty=\|a_n * w\|_\infty, \qquad n\in\mathbb{N}, \tag{2.4}$$

where $a_n * w\in(\ell_0(\mathbb{Z}))^r$ is defined by

$$a_n * w(\alpha)=\sum_{\beta\in\mathbb{Z}} a_n(\alpha-\beta)\,w(\beta), \qquad \alpha\in\mathbb{Z},$$

and for $v\in(\ell_0(\mathbb{Z}))^r$,

$$\|v\|_\infty=\max\{\|v(\alpha)\|_{\ell_\infty} : \alpha\in\mathbb{Z}\}.$$

Here for a vector $u=(u_1, ..., u_r)^T\in\mathbb{C}^r$, $\|u\|_{\ell_\infty}:=\max\{|u_j|: j=1, ..., r\}$. The formula (2.4) will be used with different w in different occurrences.

The following lemma which will be used for proving Theorem 1 gives us another way to define the sequence $\{a_n\}_{n\in\mathbb{N}}$.

LEMMA. *Let $a \in (\ell_0(\mathbb{Z}))^{r\times r}$ and the sequence $\{a_n\}_{n\in\mathbb{N}}$ be given by* (2.2). *Then*

$$a_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\beta)\, a_n(\alpha - 2^n\beta), \qquad \alpha\in\mathbb{Z}, \quad n\in\mathbb{N}. \tag{2.5}$$

Proof. We prove by induction on n. The case $n=1$ is trivial from (2.2).

Suppose that (2.5) has been verified for n. Then by (2.2) and the induction hypothesis, for $\alpha\in\mathbb{Z}$,

$$\begin{aligned} a_{n+2}(\alpha) &= \sum_{\beta\in\mathbb{Z}} \left\{ \sum_{\gamma\in\mathbb{Z}} a(\gamma)\, a_n(\beta - 2^n\gamma) \right\} a(\alpha - 2\beta) \\ &= \sum_{\gamma\in\mathbb{Z}} a(\gamma) \sum_{\beta\in\mathbb{Z}} a_n(\beta)\, a(\alpha - 2^{n+1}\gamma - 2\beta). \end{aligned}$$

It follows from (2.2) that

$$a_{n+2}(\alpha) = \sum_{\gamma\in\mathbb{Z}} a(\gamma)\, a_{n+1}(\alpha - 2^{n+1}\gamma).$$

This proves (2.5) for $n+1$, and thereby completing the induction procedure. Hence (2.5) holds for all $n\in\mathbb{N}$. ∎

We are in a position to prove Theorem 1.

Proof of Theorem 1. Suppose that $\phi = (\phi_1, ..., \phi_r)^T$ is a compactly supported continuous solution of (1.1) with $\phi(\alpha) = v(\alpha)$ for every $\alpha\in\mathbb{Z}$. Then by (2.3),

$$\phi(m/2^n) = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\, \phi(m-\alpha) = a_n * v(m), \qquad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

Taking the difference implies

$$\phi(m/2^n) - \phi((m-1)/2^n) = a_n * \nabla v(m), \qquad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

Since ϕ is compactly supported, the components are uniformly continuous. Hence

$$\|a_n * \nabla v\|_\infty \leqslant \|\phi - \phi(\cdot - 1/2^n)\|_\infty \to 0 \qquad (n\to\infty).$$

It follows from (1.4), (1.5), and (2.4) that

$$\rho_\infty(\mathscr{A}\,|_{V(\nabla v)}) < 1.$$

This completes the proof of the necessity part.

In order to prove the sufficiency part, we use the idea of Daubechies and Lagarias in [5] to construct a continuous solution.

Suppose that (2.1) holds, then for any ρ with $\rho_\infty(\mathscr{A}\,|_{V(\nabla v)})<\rho<1$, there is some constant $C>0$ such that

$$\|\mathscr{A}^n\,\nabla v\|_\infty \leqslant C\rho^n, \qquad \forall n\in\mathbb{N}.$$

By (2.4) this implies

$$\|a_n * \nabla v\|_\infty = \|\nabla(a_n * v)\|_\infty \leqslant C\rho^n, \qquad \forall n\in\mathbb{N}. \tag{2.6}$$

Let us first define the solution ϕ on dyadic points:

$$\phi(m) := v(m) \qquad \text{and} \qquad \phi(m/2^n) := a_n * v(m), \qquad \text{for} \quad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

To avoid any conflict, we observe from $A_0 v = v$ that for $m\in\mathbb{Z}$ and $n\in\mathbb{N}$,

$$\begin{aligned} a_n * v(2m) &= \sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta) \sum_{\alpha\in\mathbb{Z}} a(\alpha-2\beta)\, v(2m-\alpha) \\ &= \sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta)\, A_0 v(m-\beta) = a_{n-1} * v(m). \end{aligned}$$

This tells us that ϕ is well-defined on dyadic points, and $\phi(\alpha)=v(\alpha)$ for $\alpha\in\mathbb{Z}$. Moreover,

$$\phi((2m+1)/2^{n+1}) - \phi(m/2^n) = \nabla(a_{n+1} * v)(2m+1), \qquad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

Now we use this expression to define ϕ on nondyadic points.

Suppose that $x\in\mathbb{R}$ is not dyadic. Then x can be uniquely written as

$$x = \alpha + \sum_{j=1}^{\infty} d_j 2^{-j}, \qquad \alpha\in\mathbb{Z}, \quad d_j\in\{0, 1\}.$$

Define

$$\phi(x) = \lim_{n\to\infty} \phi\left(\alpha + \sum_{j=1}^{n} d_j 2^{-j}\right).$$

To see the existence of the limit, we observe that for $n\in\mathbb{N}$, if $d_{n+1}=0$, then

$$\phi\left(\alpha + \sum_{j=1}^{n+1} d_j 2^{-j}\right) - \phi\left(\alpha + \sum_{j=1}^{n} d_j 2^{-j}\right) = 0.$$

If $d_{n+1}=1$, then by (2.6)

$$\left\|\phi\left(\alpha+\sum_{j=1}^{n+1} d_j 2^{-j}\right)-\phi\left(\alpha+\sum_{j=1}^{n} d_j 2^{-j}\right)\right\|_{\ell_\infty} \leqslant \|\nabla(a_{n+1} * v)\|_\infty \leqslant C\rho^{n+1}.$$

Since ρ is less than 1, this shows that the limit exists and ϕ is well-defined for $x \in \mathbb{R}$. Moreover,

$$\left\|\phi(x)-\phi\left(\alpha+\sum_{j=1}^{n} d_j 2^{-j}\right)\right\|_{\ell_\infty} \leqslant \sum_{l=n+1}^{\infty} C\rho^l \leqslant C\rho^{n+1}/(1-\rho), \qquad \forall n \in \mathbb{N}.$$

This estimate is also valid for dyadic points $x=\alpha+\sum_{j=1}^{\infty} d_j 2^{-j}$ when $d_j=0$ for sufficiently large j.

To see the continuity of ϕ, we state that

$$\|\phi(x)-\phi(y)\|_{\ell_\infty} \leqslant (2C/(1-\rho)+C/\rho)\,|x-y|^{-\log_2 \rho}, \qquad \forall x \neq y \in \mathbb{R}. \qquad (2.7)$$

Let $x<y \in \mathbb{R}$. Suppose that $2^{-n-1} \leqslant |x-y| < 2^{-n}$ for certain $n \in \mathbb{N}$. There exists some $m \in \mathbb{Z}$ such that $m/2^n \leqslant y < (m+1)/2^n$. Then either $m/2^n \leqslant x < y < (m+1)/2^n$ or $(m-1)/2^n < x < m/2^n \leqslant y < (m+1)/2^n$.

In the first case,

$$\begin{aligned}\|\phi(x)-\phi(y)\|_{\ell_\infty} &\leqslant \left\|\phi(x)-\phi\left(\frac{m}{2^n}\right)\right\|_{\ell_\infty}+\left\|\phi(y)-\phi\left(\frac{m}{2^n}\right)\right\|_{\ell_\infty}\\ &\leqslant \frac{2C\rho^{n+1}}{1-\rho} \leqslant \frac{2C}{1-\rho}\,|x-y|^{-\log_2 \rho}.\end{aligned}$$

In the second case,

$$\begin{aligned}\|\phi(x)-\phi(y)\|_{\ell_\infty} \leqslant{}& \left\|\phi(x)-\phi\left(\frac{m-1}{2^n}\right)\right\|_{\ell_\infty}+\left\|\phi\left(\frac{m-1}{2^n}\right)-\phi\left(\frac{m}{2^n}\right)\right\|_{\ell_\infty}\\ &+\left\|\phi(y)-\phi\left(\frac{m}{2^n}\right)\right\|_{\ell_\infty}\\ \leqslant{}& 2C\rho^{n+1}/(1-\rho)+C\rho^n \leqslant (2C/(1-\rho)+C/\rho)\,|x-y|^{-\log_2 \rho}.\end{aligned}$$

Combining the above two cases, we know that our statement (2.7) holds true. This immediately implies the continuity of ϕ if we can prove that ϕ is compactly supported.

Suppose that supp $a \subset [N_1, N_2]$ with $N_1 < N_2$. Then it can be easily seen that for $j \leqslant N_1$, $k \geqslant N_2$, A_0 maps $(\ell[j, k])^r$ into $(\ell[(N_1+j)/2, (N_2+k)/2])^r$. Since $A_0 v = v$, we must have $v \in (\ell[N_1, N_2])^r$.

Observe that $\operatorname{supp} a_n \subset [(2^n-1)N_1, (2^n-1)N_2]$ for $n \in \mathbb{N}$. Then $\operatorname{supp}(a_n * v) \subset [2^n N_1, 2^n N_2]$. Therefore, by our construction, ϕ is supported in $[N_1, N_2]$.

Finally, we use Lemma to verify the refinement relation (1.1). Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then

$$\begin{aligned}\phi(m/2^{n+1}) &= \sum_{\alpha \in \mathbb{Z}} \left\{ \sum_{\beta \in \mathbb{Z}} a(\beta)\, a_n(\alpha - 2^n\beta) \right\} v(m-\alpha) \\ &= \sum_{\beta \in \mathbb{Z}} a(\beta)\, a_n * v(m - 2^n\beta) = \sum_{\beta \in \mathbb{Z}} a(\beta)\, \phi(m/2^n - \beta).\end{aligned}$$

Thus, the refinement relation (1.1) holds on dyadic points. By the continuity of ϕ, it holds for all $x \in \mathbb{R}$.

If ϕ is a compactly supported continuous solution of (1.1) with $\operatorname{supp} a \subset [N_1, N_2]$, then by [13], $\operatorname{supp} \phi \subset [N_1, N_2]$. Hence $v = \phi|_{\mathbb{Z}}$ is supported in $[N_1+1, N_2-1]$.

The proof of Theorem 1 is complete. ▮

The estimate (2.7) in connection with the equality $\|a_n * \nabla v\|_\infty = \|\mathscr{A}^n(\nabla v)\|_\infty = \|[\phi(\cdot/2^n) - \phi((\cdot-1)/2^n)]|_{\mathbb{Z}}\|_\infty$ provides us an exact formula for the first order Lipschitz exponent of the continuous solution.

THEOREM 2. *Let $a \in (\ell_0(\mathbb{Z}))^{r\times r}$ and $v \in (\ell_0(\mathbb{Z}))^r$ be an eigenvector of A_0 associated with eigenvalue 1. If $\rho_\infty(\mathscr{A}|_{V(\nabla v)}) < 1$, then the refinement equation (1.1) has a compactly supported continuous solution ϕ such that $\phi|_{\mathbb{Z}} = v$ and*

$$\sup\left\{\sigma > 0 : \sup_{x \neq y} \frac{\|\phi(x) - \phi(y)\|_{\ell_\infty}}{|x-y|^\sigma} < \infty\right\} = -\log_2 \rho_\infty(\mathscr{A}|_{V(\nabla v)}) > 0.$$

From our proof we can see that if a is supported in $[N_1, N_2]$, then $A_0 v = v$ is equivalent to that $(v(n))_{n=N_1}^{N_2}$ is an eigenvector of the matrix $(a(2k-n))_{k,n=N_1}^{N_2}$ associated with eigenvalue 1, and $v(n) = 0$ for $n \notin [N_1, N_2]$.

One way to construct continuous multiple refinable functions is by vector subdivision schemes, for which the uniform convergence was characterized by Jia, Riemenschneider, and Zhou in [18]. The problem settled in Theorem 1 is more general than the convergence of the subdivision scheme. Also, it does not require stability. Some other sufficient conditions for the existence of continuous solutions were presented in [2, 19].

In the scalar case ($r = 1$), uniform convergence of subdivision schemes was considered by Micchelli and Prautzsch [23], by Daubechies and Lagarias [5], by Dyn *et al.* [9], and by Jia [15]. Necessary and sufficient conditions for scalar refinement equations to have continuous solutions were given by Micchelli and Prautzsch [23], and Colella and Heil [3]. For the existence of L^p solutions, see Jia [15], and Lau and Wang [21].

Let us now apply Theorem 1 to the first step of constructing Hermite interpolants. For $j=1, ..., r$, we use e_j to denote the jth column of the $r \times r$ identity matrix. For $\beta \in \mathbb{Z}$, we use δ_β to denote the sequence given by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha = \beta, \\ 0 & \text{for} \quad \alpha \in \mathbb{Z} \backslash \{\beta\}. \end{cases}$$

For an $r \times 1$ vector $y \in \mathbb{C}^r$, $y\delta_\beta$ is the obvious element in $(\ell_0(\mathbb{Z}))^r$ given by $(y\delta_\beta)(\alpha) = y$ if $\alpha = \beta$ and $(y\delta_\beta)(\alpha) = 0$ otherwise. Also, $y \nabla \delta_\beta := y\delta_\beta - y\delta_{\beta+1} = \nabla(y\delta_\beta)$.

THEOREM 3. *Let $n_0 \in \mathbb{Z}$ and a be in $(\ell_0(\mathbb{Z}))^{r \times r}$. Then the refinement equation* (1.1) *has a compactly supported continuous solution ϕ satisfying $\phi(n) = e_1 \delta_{n_0}(n)$ for every $n \in \mathbb{Z}$ if and only if the following conditions hold*:

(a) $a(2n - n_0)\, e_1 = e_1 \delta_{n_0}(n)$ *for all* $n \in \mathbb{Z}$;

(b) $\rho_\infty(\mathscr{A} \,|_{V(e_1 \nabla \delta_{n_0})}) < 1$.

Proof. The sufficiency follows directly from Theorem 1 since by Condition (a), the sequence $e_1 \delta_{n_0} \in (\ell_0(\mathbb{Z}))^r$ is an eigenvector of A_0 associated with eigenvalue 1:

$$A_0(e_1 \delta_{n_0})(n) = a(2n - n_0)\, e_1 = e_1 \delta_{n_0}(n), \qquad n \in \mathbb{Z}.$$

To see the necessity, let ϕ be a compactly supported continuous solution of (1.1) with $\phi(n) = e_1 \delta_{n_0}(n)$ for $n \in \mathbb{Z}$. Then for $n \in \mathbb{Z}$,

$$\phi(n) = e_1 \delta_{n_0}(n) = \sum_{\beta \in \mathbb{Z}} a(\beta)\, \phi(2n - \beta) = a(2n - n_0)\, e_1.$$

This implies Condition (a) immediately. Also, $\{\phi(n)\}_{n \in \mathbb{Z}}$ is an eigenvector of A_0 associated with eigenvalue 1.

Condition (b) is an easy consequence of Theorem 1. Hence the necessity part of Theorem 3 is valid, and we have completed the proof. ∎

3. CHARACTERIZATIONS OF SMOOTHNESS AND HERMITE INTERPOLANTS

In this section we characterize multiple refinable Hermite interpolants in terms of the refinement masks. Toward this end, the optimal smoothness of continuous multiple refinable functions will be obtained without assuming stability, which extends the previous results in [2, 19, 24] where the stability was assumed.

In the scalar case ($r=1$), conditions for refinable functions to be continuously differentiable were given by Cavaretta *et al.* [1] in terms of convergence of subdivision schemes. Daubechies and Lagarias [5] used the joint spectral radius to estimate higher orders of Lipschitz exponents. Under the assumption of stability Villemoes [28] characterized the optimal smoothness in L_p ($1 \leqslant p \leqslant \infty$) by the norms of subdivision operators in l_p. A characterization in terms of joint spectral radius was presented by Jia [15]. These two approaches are equivalent, see [10, 11]. For discussions without stability, see [3, 20, 23]. For the multivariate case, we refer the reader to the work of Jia [16].

We use the generalized Lipschitz space to measure smoothness of a given function. For $y \in \mathbb{R}$ and $k \in \mathbb{N}$, the kth difference operator ∇_y^k is defined by

$$\nabla_y^k f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l f(x - ly), \qquad f \in C(\mathbb{R}).$$

For $v > 0$, let k be an integer greater than v. The *generalized Lipschitz space* Lip* v consists of those functions $f \in C(\mathbb{R})$ for which

$$\|\nabla_h^k f\|_\infty \leqslant Ch^v \qquad \forall h > 0,$$

where C is a positive constant independent of h.

By $(\text{Lip}^*\, v)^r$ we denote the linear space of all vectors $f = (f_1, ..., f_r)^T$ such that $f_1, ..., f_r \in \text{Lip}^*\, v$. The optimal smoothness of a vector $f \in (C(\mathbb{R}))^r$ is described by its *critical exponent* $v_\infty(f)$ defined by

$$v_\infty(f) := \sup\{v : f \in (\text{Lip}^*\, v)^r\}.$$

The following result is from approximation theory: For $f \in C(\mathbb{R})$ and $v > 0$, f lies in Lip* v if and only if, for some integer $k > v$, there exists a constant $C > 0$ such that

$$|\nabla_{2^{-n}}^k f(\alpha/2^n)| \leqslant C 2^{-nv} \qquad \forall n \in \mathbb{N}, \quad \alpha \in \mathbb{Z}.$$

For this result we refer the reader to the work of Ditzian [7].

With the above preliminary results, we can now state our main result on the optimal smoothness of multiple refinable functions as follows.

THEOREM 4. *Let* $a \in (\ell_0(\mathbb{Z}))^{r \times r}$, $k \in \mathbb{N}$ *and* $v \in (\ell_0(\mathbb{Z}))^r$ *be an eigenvector of* A_0 *associated with eigenvalue* 1. *Suppose that* $\phi = (\phi_1, ..., \phi_r)^T$ *is a compactly* *supported continuous solution of* (1.1) *such that* $\phi(n)=v(n)$ *for* $n\in\mathbb{Z}$. *If* $\rho_\infty(\mathscr{A}|_{V(\nabla^k v)})>(1/2)^k$, *then*

$$\nu_\infty(\phi)=-\log_2\rho_\infty(\mathscr{A}|_{V(\nabla^k v)}). \tag{3.1}$$

Proof. Suppose that ϕ is a compactly supported continuous solution of (1.1) and $\phi(\alpha)=v(\alpha)$ for all $\alpha\in\mathbb{Z}$. Then by (2.3),

$$\phi(m/2^n)=a_n * v(m), \qquad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

Hence

$$\nabla^k_{2^{-n}}\phi(m/2^n)=a_n*(\nabla^k v)(m), \qquad m\in\mathbb{Z}, \quad n\in\mathbb{N}.$$

It follows that

$$\sup_{m\in\mathbb{Z}}\|\nabla^k_{2^{-n}}\phi(m/2^n)\|_{\ell_\infty}=\|a_n*(\nabla^k v)\|_\infty, \qquad n\in\mathbb{N}.$$

Write ρ for $\rho_\infty(\mathscr{A}|_{V(\nabla^k v)})$. By (1.4) and (2.4), for $\varepsilon>0$, there exists a positive constant C such that

$$\|a_n*(\nabla^k v)\|_\infty\leqslant C(\rho+\varepsilon)^n=C(2^{-n})^\nu, \qquad \forall n\in\mathbb{N},$$

where $\nu:=-\log_2(\rho+\varepsilon)$. Therefore, for all $n\in\mathbb{N}$ and $m\in\mathbb{Z}$,

$$\|\nabla^k_{2^{-n}}\phi(m/2^n)\|_{\ell_\infty}\leqslant C(2^{-n})^\nu.$$

Since ϕ is in $(C(\mathbb{R}))^r$ and $k>-\log_2\rho>\nu$, this implies that $\phi\in(\mathrm{Lip}^*\nu)^r$. Thus,

$$\nu_\infty(\phi)\geqslant-\log_2(\rho+\varepsilon).$$

But $\varepsilon>0$ can be arbitrarily small; hence

$$\nu_\infty(\phi)\geqslant-\log_2\rho.$$

Now we show that $\nu_\infty(\phi)\leqslant-\log_2\rho$. Suppose to the contrary that $\nu_\infty(\phi)>-\log_2\rho$. Then there is some $\nu\in(-\log_2\rho,k)$ such that $\phi\in(\mathrm{Lip}^*\,\nu)^r$. Hence for some constant $C>0$,

$$\sup_{m\in\mathbb{Z}}\|\nabla^k_{2^{-n}}\phi(m/2^n)\|_{\ell_\infty}=\|a_n*(\nabla^k v)\|_\infty\leqslant C2^{-n\nu}, \qquad \forall n\in\mathbb{N}.$$

By (1.4) and (2.4), this tells

$$\rho_\infty(\mathscr{A}|_{V(\nabla^k v)})=\lim_{n\to\infty}\|\mathscr{A}^n(\nabla^k v)\|_\infty^{1/n}\leqslant 2^{-\nu}.$$

Thus,

$$\nu \leqslant -\log_2 \rho,$$

which is a contradiction. Therefore, we obtain the desired result $\nu_\infty(\phi) \leqslant -\log_2 \rho$. ∎

Remark. If $1 \leqslant k \leqslant -\log_2 \rho_\infty(\mathscr{A}|_{V(\nabla^k v)})$, then our proof of the necessity part shows that $\phi \in (\mathrm{Lip}^*\,(k-\varepsilon))^r$ for any $\varepsilon > 0$. Hence

$$\nu_\infty(\phi) \geqslant k.$$

This implies that $\phi \in (C^{k-1}(\mathbb{R}))^r$. Taking derivatives in (1.1),

$$\phi^{(k-1)}(x) = 2^{k-1} \sum_{\alpha \in \mathbb{Z}} a(\alpha)\, \phi^{(k-1)}(2x-\alpha).$$

It follows that $\phi^{(k-1)}|_{\mathbb{Z}} \neq 0$, since otherwise, $\phi^{(k-1)} \equiv 0$ and ϕ is a vector of polynomials on $\mathbb{R}$, which would be a contradiction to the compact support of ϕ.

Thus, taking values at integers, $\{\phi^{(k-1)}(\alpha)\}_{\alpha \in \mathbb{Z}}$ is an eigenvector of A_0 associated with eigenvalue 2^{1-k}. Restricting to the interval $[N_1, N_2-1]$, $\{\phi^{(k-1)}(\alpha)\}_{\alpha=N_1}^{N_2-1} \neq 0$ is an eigenvector of the blockmatrix $A_0|_{[N_1, N_2-1]} := (a(2\alpha-\beta))_{\alpha,\beta=N_1}^{N_2-1}$ associated with eigenvalue 2^{1-k}.

Note that the order of $A_0|_{[N_1, N_2-1]}$ is $(N_2-N_1)r$. One of $\{2^{1-k}\}_{1 \leqslant k \leqslant (N_2-N_1)r+1}$ is not its eigenvalue. Therefore, $k \leqslant (N_2-N_1)r$ if ϕ is in $(C^{k-1}(\mathbb{R}))^r$. This shows that

$$-\log_2 \rho_\infty(\mathscr{A}|_{V(\nabla^k v)}) < (N_2-N_1)\, r+1 \qquad \forall k \in \mathbb{N}.$$

Thus $\rho_\infty(\mathscr{A}|_{V(\nabla^k v)}) > (1/2)^k$ always holds if the integer k is greater than $(N_2-N_1)r$.

Let us now turn to our main result on the characterization of multiple refinable Hermite interpolants. For a compactly supported continuous function f on $\mathbb{R}$, and $l \in \mathbb{N}$, $0<\varepsilon<1$, it is well-known that $f \in \mathrm{Lip}^*\,(l+\varepsilon)$ if and only if $f^{(l)} \in \mathrm{Lip}^*\,\varepsilon$. Moreover, if $\phi \in (C^l(\mathbb{R}))^r$ is a compactly supported multiple refinable function with its shifts linearly independent, then by [30], $\phi \in (\mathrm{Lip}^*(l+\lambda))^r$ for some $\lambda > 0$. To see this fact, the following observation plays an important role: If $f \in C^l(\mathbb{R})$ is compactly supported, then for any integer $k > l$, $\|\nabla_h^k f\|_\infty = o(h^l)$ $(h>0)$. This in connection with our previous results on existence and smoothness provides our main result on multiple refinable Hermite interpolants. Denote $\mathrm{diag}\{1, 1/2, ..., (1/2)^{r-1}\}$ as the diagonal $r \times r$ matrix with the diagonal elements $1, 1/2, ..., (1/2)^{r-1}$. For $v \in (\ell_0(\mathbb{Z}))^r$, let $v|_{[s,t]} \in (\ell([s,t]))^r$ be the restriction of v onto $[s,t]$.

THEOREM 5. *Let $n_0 \in \mathbb{Z}$, a be in $(\ell_0(\mathbb{Z}))^{r\times r}$ with* $\operatorname{supp} a = [N_1, N_2]$ *for two integers $N_1 < N_2$. Then the refinement equation* (1.1) *has a compactly supported solution ϕ of multiple refinable Hermite interpolant at n_0*, i.e.,

$$\phi \in (C^{r-1}(\mathbb{R}))^r \qquad \text{and} \qquad \phi^{(m-1)}(n) = e_m \delta_{n_0}(n), \qquad m = 1, ..., r, \quad n \in \mathbb{Z}, \tag{3.2}$$

if and only if the following conditions are satisfied:

(a) $a(2n - n_0) = \operatorname{diag}\{1, 1/2, ..., (1/2)^{r-1}\}\, \delta_{n_0}(n)$ *for all* $n \in \mathbb{Z}$;

(b) $\rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla \delta_{n_0})}) < 1$;

(c) $\rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) < 2^{1-r}$ *for some integer* $k \geqslant r$;

(d) $\lim_{n\to\infty} 2^{nm} (A_0|_{[N_1, T_m]})^n\, (e_1 \nabla^m \delta_{n_0}|_{[N_1, T_m]}) = e_{m+1} \delta_{n_0}|_{[N_1, T_m]}$ *for* $m = 1, ..., r-1$ *and* $T_m := \max\{N_2 - 1, n_0 + m\}$.

Proof. Suppose that ϕ is a compactly supported solution of (1.1) such that (3.2) holds. Then by Theorem 3, Condition (b) is true.

Taking derivatives at integers in (1.1), we know from (3.2) that for $m = 1, ..., r$ and $n \in \mathbb{Z}$,

$$e_m \delta_{n_0}(n) = \phi^{(m-1)}(n) = 2^{m-1} \sum_{\beta \in \mathbb{Z}} a(\beta)\, \phi^{(m-1)}(2n - \beta) = 2^{m-1} a(2n - n_0)\, e_m.$$

That means, for $m = 1, ..., r$ and $n \in \mathbb{Z}$,

$$a(2n - n_0)\, e_m = (1/2)^{m-1}\, e_m \delta_{n_0}(n).$$

Hence Condition (a) holds as well. Note that $v = \phi|_{\mathbb{Z}} = e_1 \delta_{n_0}$ is an eigenvector of A_0 associated with eigenvalue 1.

It is easily seen from the Hermite interpolating condition (3.2) that the shifts of $\phi_1, ..., \phi_r$ are linearly independent, that is,

$$\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}} b_j(\alpha)\, \phi_j(\cdot - \alpha) = 0 \Rightarrow b_j(\alpha) = 0 \qquad \forall\, \alpha \in \mathbb{Z}, \quad j = 1, ..., r.$$

In fact,

$$\left[\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}} b_j(\alpha)\, \phi_j(\cdot - \alpha) \right]^{(m-1)} (\beta) = b_m(\beta - n_0)$$

for any $\beta \in \mathbb{Z}$ and $m = 1, ..., r$.

Thus, we know from [30, Theorem 3] that $\phi^{(r-1)} \in (\mathrm{Lip}^*\, \lambda)^r$ for some $\lambda > 0$. Hence $\nu_\infty(\phi) > r - 1$.

For any integer $k \geqslant r$, either

$$-\log_2 \rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) \geqslant k > r-1;$$

or $-\log_2 \rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) < k$, which implies by Theorem 4 that

$$-\log_2 \rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) = \nu_\infty(\phi) > r-1.$$

In both cases,

$$\rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) < 2^{1-r}.$$

Hence Condition (c) is valid.

To show the last condition, recall the definition of the linear operator A_0 and the sequence a_n. By (2.3),

$$\phi(\alpha/2^n) = a_n * (e_1 \delta_{n_0})(\alpha), \qquad \alpha \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

It follows that for $m = 1, ..., r-1$, $\alpha \in \mathbb{Z}$,

$$\nabla^m_{2^{-n}} \phi(\alpha) = a_n * (e_1 \nabla^m \delta_{n_0})(2^n \alpha) = \sum_{\beta \in \mathbb{Z}} a_n(\beta)\, e_1 \nabla^m \delta_{n_0}(2^n \alpha - \beta).$$

Thus, by induction on n, for $\alpha \in \mathbb{Z}$,

$$\begin{aligned} \nabla^m_{2^{-n}} \phi(\alpha) &= \sum_{\beta \in \mathbb{Z}} \left[\sum_{\gamma \in \mathbb{Z}} a_{n-1}(\gamma)\, a(\beta - 2\gamma) \right] e_1 \nabla^m \delta_{n_0}(2^n \alpha - \beta) \\ &= \sum_{\gamma \in \mathbb{Z}} a_{n-1}(\gamma)\, A_0(e_1 \nabla^m \delta_{n_0})(2^{n-1}\alpha - \gamma) = \cdots \\ &= A_0^n(e_1 \nabla^m \delta_{n_0})(\alpha). \end{aligned}$$

Therefore, for $m = 1, ..., r-1$ and $\alpha \in \mathbb{Z}$,

$$\lim_{n \to \infty} 2^{nm}(A_0^n(e_1 \nabla^m \delta_{n_0}))(\alpha) = \lim_{n \to \infty} 2^{nm} \nabla^m_{2^{-n}} \phi(\alpha) = \phi^{(m)}(\alpha) = e_{m+1} \delta_{n_0}(\alpha).$$

Since $(\ell([N_1, T_m]))^r$ is invariate under A_0, Condition (d) follows. The proof of the necessity is complete.

To show the sufficiency, suppose that all the four conditions hold. Then the first two conditions in connection with Theorem 3 implies that the refinement equation (1.1) has a compactly supported continuous solution ϕ with $\phi(n) = e_1 \delta_{n_0}(n)$ for $n \in \mathbb{Z}$.

To see that $\phi \in (C^{r-1}(\mathbb{R}))^r$, use Condition (c). If $-\log_2 \rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) < k$, then Theorem 4 tells that $\nu_\infty(\phi) > r-1$, hence $\phi \in (C^{r-1}(\mathbb{R}))^r$. If $-\log_2 \rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^k \delta_{n_0})}) \geqslant k$, then Remark after the proof of Theorem 4 tells $v_\infty(\phi) \geqslant k > r-1$, which implies $\phi \in (C^{r-1}(\mathbb{R}))^r$ again. Combining these two cases, we conclude that ϕ is in $(C^{r-1}(\mathbb{R}))^r$.

It remains to prove (3.2) for $m = 2, ..., r$ and $N_1 + 1 \leqslant \alpha \leqslant N_2 - 1$, since ϕ is supported in $[N_1, N_2]$. But the same procedure as in the proof of the necessity part shows that for $m = 1, ..., r-1$, and $\alpha = N_1 + 1, ..., N_2 - 1$,

$$\phi^{(m)}(\alpha) = \lim_{n \to \infty} 2^{nm} \nabla^m_{2^{-n}} \phi(\alpha) = \lim_{n \to \infty} 2^{nm} (A_0^n (e_1 \nabla^m \delta_{n_0}))(\alpha)$$

$$= \lim_{n \to \infty} 2^{nm} (A_0 |_{[N_1, T_m]})^n (e_1 \nabla^m \delta_{n_0} |_{[N_1, T_m]})(\alpha).$$

This in connection with Condition (d) tells us that $\phi^{(m)}(\alpha) = e_{m+1} \delta_{n_0}(\alpha)$, for $m = 1, ..., r-1$ and $\alpha = N_1 + 1, ..., N_2 - 1$, thereby completing the proof of the sufficiency. ∎

From the above proof we can see that under the conditions of Theorem 5, for any integer $k \geqslant r$ we have $\rho_\infty(\mathscr{A} |_{V(e_1 \nabla^k \delta_{n_0})}) < 2^{1-r}$.

Condition (d) in Theorem 5 can be easily checked using the Jordan canonical form of A_0 restricted to the interval $[N_1, \max\{N_2 - 1, n_0 + m\}]$.

Note that for $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and $v \in (\ell_0(\mathbb{Z}))^r$, $(\nabla^k v)(\cdot - \alpha) = \sum_{l=0}^k \binom{k}{l} (-1)^l v(\cdot - \alpha - l) = \nabla^k (v(\cdot - \alpha))$. It follows that for $k \in \mathbb{N}$, n_0, $\beta \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\|a_n * (e_1 \nabla^k \delta_{n_0})\|_\infty = \|a_n * (e_1 \nabla^k \delta_{n_0})(\cdot - \beta + n_0)\|_\infty = \|a_n * (e_1 \nabla^k \delta_\beta)\|_\infty.$$

This in connection with (1.4) and (2.4) implies that

$$\rho_\infty(\mathscr{A} |_{V(e_1 \nabla^k \delta_{n_0})}) = \rho_\infty(\mathscr{A} |_{V(e_1 \nabla^k \delta_\beta)}), \qquad k \in \mathbb{N}, \quad n_0, \beta \in \mathbb{Z}. \tag{3.3}$$

Condition (a) in Theorem 5 tells us that $n_0 - N_1$, $N_2 - n_0 \in -1 + 2\mathbb{N}$.

The proof of Theorem 5 shows that the Hermite interpolating condition (3.2) implies the linear independence, and hence the stability of the shifts of ϕ. From Dahmen and Micchelli [4], it follows that the matrix $\sum_{n \in \mathbb{Z}} a(n)/2$ has a simple eigenvalue 1 and all its other eigenvalues are less than 1 in modulus. Under this form, the stability implies the uniform convergence of the subdivision scheme, see [18]. Consequently, if one is only interested in Hermite interpolant solutions, the general results in Section 2, Theorems 1 and 2, can be replaced by the characterization of the convergence of the subdivision scheme given in [18]. However, Theorem 1 is of independent interest for the existenece of continuous multiple refinable functions without assuming stability.

4. EXAMPLES

In this section we present examples to illustrate the general theory. In particular, we show how to reduce the orders of matrices for the computation of joint spectral radii by restricting to invariant subspaces. The advantages of this approach can be seen from the examples treated in [19] including the orthogonal multiple wavelets of DGHM [8].

Another way to characterize multiple refinable Hermite interpolants is by the factorization introduced by Plonka [25]. In fact, under the assumption of stability, a compactly supported continuous solution of (1.1) is in $(C^{r-1}(\mathbb{R}))^r$ if and only if the vector subdivision scheme associated with the new mask after $r-1$ times factorizations converges uniformly, see e.g., [2, 24]. However, this approach involves the computation of joint spectral radii of matrices which have higher orders due to the following reasons: Firstly, the factorization in the vector case often enlarges the supports of the refinement masks; Secondly, the dimension of the subspaces in checking the convergence of vector subdivision schemes is more than that of the common invariant subspaces containing some $v \in (\ell_0(\mathbb{Z}))^r$.

Let A be a linear operator on a linear space V with $\{v_1, ..., v_s\}$ as its basis. Suppose $Av_k = \sum_{j=1}^{s} a_{jk} v_j$ for $1 \leqslant k \leqslant s$. Then the matrix $(a_{jk})_{1 \leqslant j, k \leqslant s}$ is said to be the matrix representation of A. This matrix representation can be regarded as a linear operator on $\mathbb{C}^s$. In this way, the joint spectral radius of a finite multiset of linear operators on V is equal to the joint spectral radius of their matrix representations as a multiset of linear operators on $\mathbb{C}^s$.

Suppose $\mathscr{A} = \{A_1, ..., A_m\}$ and each A_j is a block triangular matrix:

$$A_j = \begin{pmatrix} E_j & G_j \\ 0 & F_j \end{pmatrix}, \qquad j = 1, ..., m,$$

where $E_1, ..., E_m$ are square matrices of the same size, and so are $F_1, ..., F_m$. It was proved in [18, Lemma 4.2] that

$$\rho_\infty(A_1, ..., A_m) = \max\{\rho_\infty(E_1, ..., E_m), \rho_\infty(F_1, ..., F_m)\}. \tag{4.1}$$

Let us turn to our first example of multiple refinable Hermite interpolant.

Example 1. *Let $a \in (\ell_0(\mathbb{Z}))^{2\times 2}$ be given by*

$$a(-3) = \begin{bmatrix} 1/128 & 1/384 \\ 0 & 0 \end{bmatrix}, \qquad a(-1) = \begin{bmatrix} 63/128 & 99/128 \\ -9/64 & -9/64 \end{bmatrix},$$

$$a(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \qquad a(1) = \begin{bmatrix} 63/128 & -99/128 \\ 9/64 & -9/64 \end{bmatrix},$$

$$a(3) = \begin{bmatrix} 1/128 & -1/384 \\ 0 & 0 \end{bmatrix}, \qquad a(n) = 0 \qquad \forall n \neq \pm 3, \pm 1, 0.$$

Then the refinement equation (1.1) *associated with this mask has a compactly supported solution* ϕ *of multiple refinable Hermite interpolant at the origin. Moreover,* $\phi \in (C^2(\mathbb{R}))^2$ *and*

$$2.3548951 < \nu_\infty(\phi) \leqslant 4 - \log_2 3 \approx 2.4150375.$$

Proof. Let us verify the conditions in Theorem 5 for the refinement mask here to draw the first conclusion.

Set $n_0 = 0$, $N_1 = -3$, $N_2 = 3$. Obviously, Condition (a) is satisfied.

To show Condition (c), by (3.3) we only need to prove

$$\rho_\infty(\mathscr{A}|_{V(e_1 \nabla^3 \delta_{-3})}) < 1/2.$$

Toward this end, let $w := e_1 \nabla^3 \delta_{-3}$. A simple computation by Maple tells us that the minimal common invariant subspace W under A_0 and A_1 containing w has a basis $\{w, A_0 w, A_1 w, A_0^2 w, A_1 A_0 w, A_1^2 w, A_0^3 w\}$. Denote the matrix representations of the operators A_0 and A_1 under this basis as B_0 and B_1, respectively. Then $\rho_\infty(\mathscr{A}|_{V(e_1 \nabla^3 \delta_{-3})}) = \rho_\infty(\mathscr{A}|_W) = \rho_\infty(\{B_0, B_1\})$.

Associated with the eigenvalues 1/8, 1/16, 1/32, 0, 3/16, 3/64 and 1/128, respectively, the matrix B_1 has eigenvectors w_1, w_2, w_3, w_4, w_5, w_6 and w_7. Here each vector $w_j \in \mathbb{C}^7$, $j = 1, \ldots, 7$, is chosen to have either the first component 1 or the first two components 0 and -1. Under the basis $B(W) := \{w_1, w_2, w_3, w_4, \frac{1}{8} w_5, w_6, 2w_7\}$, the matrices B_0 and B_1 have the matrix representations

$$\begin{bmatrix} 1/8 & 0 & 0 & 0 & \cdots & 0 \\ * & 1/16 & 0 & 0 & \cdots & 0 \\ * & * & 1/32 & 0 & \cdots & 0 \\ * & * & * & & & \\ \vdots & \vdots & \vdots & & C_0 & \\ * & * & * & & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/8 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1/16 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1/32 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & C_1 & \\ 0 & 0 & 0 & & & \end{bmatrix},$$

respectively, where

$$C_0 = \begin{bmatrix} -3/32 & 0 & 0 & 713/2048 \\ -5/713 & 13/736 & 3/368 & 477/29440 \\ 5/186 & -1/48 & -1/320 & -13/200 \\ -315/2852 & 15/736 & -9/736 & 4731/14720 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3/16 & 0 & 0 \\ 0 & 0 & 3/64 & 0 \\ 0 & 0 & 0 & 1/128 \end{bmatrix}.$$

It is easily seen that for each C_0 and C_1, the norm on $(\mathbb{C}^4, \|\cdot\|_{\ell_\infty})$ is less than 1/2. This in connection with (4.1) tells that

$$\rho_\infty(\mathscr{A}\,|_{V(e_1 \nabla^3 \delta_{-3})}) = \rho_\infty(\{B_0, B_1\}) = \max\{1/8, \rho_\infty(\{C_0, C_1\})\} < 1/2.$$

Hence Condition (c) holds.

Now we apply the conclusion for Condition (c) to the proof of Condition (b). Let $v_1 = e_1 \nabla \delta_{-3}$, $v_2 = e_1 \nabla^2 \delta_{-3}$. Then the minimal common invariant subspace under A_0 and A_1 containing v_1 and v_2 is contained in $\text{span}\{v_1, v_2\} + W$. Under the basis $\{v_1, v_2, B(W)\}$, the operators A_0 and A_1 restricted to $\text{span}\{v_1, v_2\} + W$ have matrix representations

$$A_0 = \begin{bmatrix} 1/2 & 0 & 0\cdots 0 \\ * & 1/4 & 0\cdots 0 \\ * & * & \\ \vdots & \vdots & A_0|_W \\ * & * & \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 1/2 & 0 & 0\cdots 0 \\ * & 1/4 & 0\cdots 0 \\ * & * & \\ \vdots & \vdots & A_1|_W \\ * & * & \end{bmatrix}.$$

It follows from (3.3) and (4.1) that

$$\begin{aligned} \rho_\infty(\mathscr{A}|_{V(e_1 \nabla \delta_0)}) = \rho_\infty(\mathscr{A}|_{V(e_1 \nabla \delta_{-3})}) &\leqslant \rho_\infty(\mathscr{A}|_{\text{span}\{v_1, v_2\} + W})\} \\ &= \max\{\tfrac{1}{2}, \rho_\infty(\mathscr{A}|_W)\} = \tfrac{1}{2}. \end{aligned}$$

Thus, Condition (b) holds true.

For the last condition, observe that $\text{span}\{e_2 \delta_0, e_1 \nabla \delta_0\} + W$ is invariant under A_0. Moreover, $A_0(e_2\delta_0) = \frac{1}{2} e_2 \delta_0$ and $A_0(e_2\delta_0 - e_1 \nabla \delta_0) - \frac{1}{4}(e_2\delta_0 - e_1 \nabla \delta_0) \in W$. However, $\rho(A_0|_W) \leqslant \rho_\infty(\mathscr{A}|_W) < 1/2$. Therefore,

$$\lim_{n\to\infty} 2^n(A_0^n(e_1 \nabla \delta_0)) = \lim_{n\to\infty} 2^n(A_0^n(e_2\delta_0) - A_0^n(e_2\delta_0 - e_1 \nabla \delta_0)) = e_2 \delta_0.$$

This completes the verification of all the conditions in Theorem 5 for the refinement mask here, thereby proving the first conclusion.

To see the second conclusion, we apply Theorem 4. Recall from the above proof and (3.3) that

$$\rho_\infty(\mathscr{A}|_{V(e_1\nabla^3\delta_0)}) = \rho_\infty(\{B_0, B_1\}) = \max\{1/8, \rho_\infty(\{C_0, C_1\})\}.$$

Also, C_1 has an eigenvalue 3/16. Hence

$$\rho_\infty(\mathscr{A}|_{V(e_1\nabla^3\delta_0)}) \geqslant \rho_\infty(\{C_0, C_1\}) \geqslant \rho(C_1) \geqslant 3/16.$$

It follows that

$$\rho_\infty(\mathscr{A}|_{V(e_1\nabla^3\delta_0)}) > 1/8.$$

Thus, by Theorem 4,

$$\begin{aligned} \nu_\infty(\phi) &= -\log_2 \rho_\infty(\mathscr{A}|_{V(e_1\nabla^3\delta_0)}) = -\log_2 \rho_\infty(\{C_0, C_1\}) \\ &\leqslant -\log_2(3/16) = 4 - \log_2 3. \end{aligned}$$

An upper bound for $\rho_\infty(\{C_0, C_1\})$ can be given by

$$\rho_\infty(\{C_0, C_1\}) \leqslant \|\{C_0, C_1\}^n\|_\infty^{1/n}, \qquad n \in \mathbb{N}.$$

Choose $n = 32$ and the norm of 4×4 matrices as the norm on $(\mathbb{C}^4, \|\cdot\|_{\ell_\infty})$. We use Maple and obtain the estimate

$$\rho_\infty(\{C_0, C_1\}) \leqslant \|\{C_0, C_1\}^{32}\|_\infty^{1/32} \leqslant 3.796195652 \times (3/16)^{32}.$$

Hence

$$\nu_\infty(\phi) \geqslant 2.3548951.$$

This proves all the statements of Example 1. ∎

If we choose $k = 5$, then $\dim V(e_1\nabla^5\delta_{-3}) = 5$, and $\rho_\infty(\mathscr{A}|_{V(e_1\nabla^5\delta_{-3})})$ is reduced into the joint spectral radius of two 5×5 matrices. Example 1 shows that we can usually reduce the order of matrices for the computation of joint spectral radius by restricting to subspaces.

The next example was introduced by Jia, Riemenschneider and Zhou in [17] for the investigation of accuracy. Under the assumption that $-3/4 < st < 1/4$, we have considered the convergence of vector subdivision schemes in [18], the L_p-optimal smoothness in [19], and the Hermite interpolating property in [30]. Here without this assumption on the parameters s and t, we apply our general theory to the study of existence, smoothness analysis and the Hermite interpolating property of continuous solutions.

EXAMPLE 2. *Let $a \in (\ell_0(\mathbb{Z}))^{2\times 2}$ be supported in $[0, 2]$ and given by*

$$a(0) = \begin{bmatrix} 1/2 & s/2 \\ t & 1/4 + 2st \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \text{and} \quad a(2) = \begin{bmatrix} 1/2 & -s/2 \\ -t & 1/4 + 2st \end{bmatrix},$$

where s and t are real parameters. Then the refinement equation (1.1) *associated with this mask has a nonzero compactly supported continuous solution ϕ if and only if $-3/4 < st < 1/4$. In this case, for the optimal smoothness,*

$$\nu_\infty(\phi) = \begin{cases} 2 & \text{if } \; |st + 1/4| \leqslant 1/8, \\ -\log_2 |\frac{1}{2} + 2st| & \text{if } \; 1/8 < |st + 1/4| < 1/2. \end{cases} \tag{4.2}$$

Finally, the refinement equation has a solution of Hermite interpolant if and only if $t = -1/8$ and $0 < s < 4$.

Proof. For the existence of continuous solutions, we apply Theorem 1. Since all the eigenvectors v of A_0 associated with eigenvalue 1 with $\operatorname{supp} v = \{1\}$ have the form $v = ce_1\delta_1$ where $c \neq 0$, by Theorem 1 the corresponding refinement equation has a nonzero compactly supported continuous solution if and only if $\rho_\infty(\mathscr{A}|_{V(e_1 \nabla\delta_1)}) < 1$. By (3.3), this is equivalent to $\rho_\infty(\mathscr{A}|_{V(e_1 \nabla\delta_0)}) < 1$.

Let

$$v_1 = \begin{bmatrix} 1 \\ 4t \end{bmatrix} \delta_0 + \begin{bmatrix} -1 \\ 4t \end{bmatrix} \delta_1, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\delta_0 - \delta_1), \qquad \text{and}$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\delta_0 - \delta_1) = e_1 \nabla\delta_0.$$

Then

$$[A_j v_1 \quad A_j v_2 \quad A_j v_3] = [v_1 \quad v_2 \quad v_3]\, B_j, \qquad j = 0, 1,$$

where

$$B_0 = \begin{bmatrix} 1/2 + 2st & s/2 & 0 \\ 0 & 1/4 & t \\ 0 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 1/2 + 2st & -s/2 & 0 \\ 0 & 1/4 & -t \\ 0 & 0 & 1/2 \end{bmatrix}.$$

If $st \neq 0$, then $V(e_1 \nabla\delta_0) = \operatorname{span}\{v_1, v_2, v_3\}$, and by (4.1),

$$\rho_\infty(\mathscr{A}|_{V(e_1 \nabla\delta_0)}) = \max\{|1/2 + 2st|, 1/2\}.$$

Thus, $\rho_\infty(\mathscr{A}|_{V(e_1 \nabla\delta_0)}) < 1$ if and only if $|1/2 + 2st| < 1$, i.e., $-3/4 < st < 1/4$.

If $t=0$, then $V(e_1\nabla\delta_0)=\text{span}\{v_3\}$, and $\rho_\infty(\mathcal{A}|_{V(e_1\nabla\delta_0)})=1/2<1$. In this case, $\phi=c(\varphi,0)^T$, where φ is the hat function supported in $[0,2]$ defined by $\varphi(x)=x$ for $0\leqslant x\leqslant 1$ and $\varphi(x)=2-x$ for $1<x\leqslant 2$.

If $t\neq 0$ and $s=0$, then $V(e_1\nabla\delta_0)=\text{span}\{v_2,v_3\}$, and by (4.1), $\rho_\infty(\mathcal{A}|_{V(e_1\nabla\delta_0)})=1/2<1$.

Combining all these three cases, we know that the refinement equation with this mask has a nonzero compactly supported continuous solution ϕ if and only if $-3/4<st<1/4$.

In what follows we assume $-3/4<st<1/4$.

To estimate the optimal smoothness of ϕ, we apply Theorem 4. By (3.1) and (3.3), $\nu_\infty(\phi)=-\log_2\rho_\infty(\mathcal{A}|_{V(e_1\nabla^3\delta_0)})$ provided $\rho_\infty(\mathcal{A}|_{V(e_1\nabla^3\delta_0)})>1/8$.

Choose

$$w_1=\begin{bmatrix}1\\4t\end{bmatrix}\delta_0+\begin{bmatrix}-1\\4t\end{bmatrix}\delta_1,\qquad w_2=\begin{bmatrix}1\\4t\end{bmatrix}\delta_1+\begin{bmatrix}-1\\4t\end{bmatrix}\delta_2,$$

$$w_3=e_2\nabla^2\delta_0,\qquad w_4=e_1\nabla^3\delta_0.$$

Then

$$[A_jw_1,A_jw_2,A_jw_3,A_jw_4]=[w_1,w_2,w_3,w_4]\,D_j,\qquad j=0,1,$$

where

$$D_0=\begin{bmatrix}\frac{1}{2}+2st & 0 & \frac{s}{2} & \frac{1}{2}\\ 0 & \frac{1}{2}+2st & \frac{s}{2} & -\frac{1}{2}\\ 0 & 0 & \frac{1}{4} & -t\\ 0 & 0 & 0 & 0\end{bmatrix}\quad\text{and}$$

$$D_1=\begin{bmatrix}\frac{1}{2}+2st & \frac{1}{2}+2st & -s & -\frac{1}{2}\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & -t\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

If $st\neq 0$, then $V(e_1\nabla^3\delta_0)=\text{span}\{w_1,w_2,w_3,w_4\}$. Hence

$$\rho_\infty(\mathcal{A}|_{V(e_1\nabla^3\delta_0)})=\max\{|1/2+2st|,1/4\}>1/8.$$

Therefore, $\rho_\infty(\mathscr{A}|_{V(e_1\,\nabla^3\delta_0)}) > 1/8$ and

$$\begin{aligned} v_\infty(\phi) &= -\log_2 \max\{|1/2+2st|,\, 1/4\} \\ &= \begin{cases} 2 & \text{if} \quad |st+1/4| \leqslant 1/8, \\ -\log_2 |\frac{1}{2}+2st| & \text{if} \quad 1/8 < |st+1/4| < 1/2. \end{cases} \end{aligned}$$

If $t=0$, then $V(e_1\,\nabla^3\delta_0) = \text{span}\{w_1 - w_2,\, w_4\}$, and

$$\rho_\infty(\mathscr{A}|_{V(e_1\,\nabla^3\delta_0)}) = |1/2+2st| = 1/2 > 1/8.$$

Hence $v_\infty(\phi) = 1$.

If $t \neq 0$ and $s=0$, then $V(e_1\,\nabla^3\delta_0) = \text{span}\{w_1 - w_2,\, w_3,\, w_4\}$, and

$$\rho_\infty(\mathscr{A}|_{V(e_1\,\nabla^3\delta_0)}) = 1/2 > 1/8.$$

Again, $v_\infty(\phi) = 1$.

Thus, in all the cases, (4.2) holds true, thereby obtaining the optimal smoothness.

Our final task is to check the conditions for the Hermite interpolation. First, the solution ϕ should be chosen such that $\phi(1) = e_1$. Such a solution is unique when $-3/4 < st < 1/4$ and $v = e_1\delta_1$. Condition (a) is trivial.

By the first two parts on existence and smoothness with $k=3$, Conditions (b) and (c) of Theorem 5 hold true if and only if $|1/2+2st| < 1/2$, i.e., $-1/2 < st < 0$.

It remains to consider the last condition. By a simple computation, for $u_1 = e_1\,\nabla\delta_1 - 4A_0(e_1\,\nabla\delta_1) + w_2$,

$$A_0 u_1 = \tfrac{1}{2} u_1.$$

For $u_2 = e_1\,\nabla\delta_1 - 2A_0(e_1\,\nabla\delta_1)$,

$$A_0 u_2 - \tfrac{1}{4} u_2 \in V(e_1\,\nabla^3\delta_0).$$

Therefore, from the known fact $\rho_\infty(\mathscr{A}|_{V(e_1\,\nabla^3\delta_0)}) < 1/2$, we obtain that

$$\lim_{n\to\infty} 2^n(A_0^n(e_1\,\nabla\delta_1)) = \lim_{n\to\infty} 2^n A_0^n(2u_2 - u_1 + w_2) = -u_1.$$

Thus, Condition (d) in Theorem 5 holds here if and only if $-u_1(1) = e_2$. But

$$u_1(1) = e_1 - 4\sum_{\beta\in\mathbb{Z}} a(2-\beta)\, e_1\,\nabla\delta_1(\beta) + (1,\, 4t)^T = (0,\, 8t)^T.$$

Hence, Condition (d) in Theorem 5 is valid for our mask if and only if $t = -1/8$.

Combining all the above discussion, we conclude that the refinement equation has a solution of Hermite interpolant if and only if $t = -1/8$ and $0 < s < 4$. ▮

The special case $s = 3/2$ and $t = -1/8$ was discussed by Heil *et al.* [14]. In this case, ϕ can be solved explicitly as

$$\phi_1(x) = \begin{cases} x^2(-2x+3) & \text{for } 0 \leqslant x \leqslant 1, \\ (2-x)^2\,(2x-1) & \text{for } 1 < x \leqslant 2, \\ 0 & \text{for } x \in \mathbb{R} \backslash [0,2], \end{cases}$$

and

$$\phi_2(x) = \begin{cases} x^2(x-1) & \text{for } 0 \leqslant x \leqslant 1, \\ (2-x)^2\,(x-1) & \text{for } 1 < x \leqslant 2, \\ 0 & \text{for } x \in \mathbb{R} \backslash [0,2]. \end{cases}$$

It is evident that $v_\infty(\phi) = 2$, and ϕ is an Hermite interpolant.

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REFERENCES

1. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary subdivision, *Mem. Amer. Math. Soc.* **93** (1991).
2. A. Cohen, I. Daubechies, and G. Plonka, Regularity of refinable function vectors, *J. Fourier Anal. Appl.* **3** (1997), 295–324.
3. D. Colella and C. Heil, Characterizations of scaling functions: Continuous solutions, *SIAM J. Matrix Anal. Appl.* **15** (1994), 496–518.
4. W. Dahmen and C. A. Micchelli, Biorthogonal wavelet expansions, *Constr. Approx.* **13** (1997), 293–328.
5. I. Daubechies and J. Lagarias, Two-scale difference equations. II. Local regularity, infinite products of matrices, and fractals, *SIAM J. Math. Anal.* **23** (1992), 1031–1079.
6. G. Deslauriers and S. Dubuc, Symmetric iterative interpolation processes, *Constr. Approx.* **5** (1989), 49–68.
7. Z. Ditzian, Moduli of smoothness using discrete data, *J. Approx. Theory* **49** (1987), 115–129.
8. G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Construction of orthogonal wavelets using fractal interpolation functions, *SIAM J. Math. Anal.* **27** (1996), 1158–1192.
9. N. Dyn, J. A. Gregory, and D. Levin, Analysis of uniform binary subdivision schemes for curve design, *Constr. Approx.* **7** (1991), 127–147.

10. T. N. T. Goodman, R. Q. Jia, and C. A. Micchelli, On the spectral radius of a bi-infinite periodic and slanted matrix, *Southeast Asian Bull. Math.* **22** (1998), 115–134.
11. T. N. T. Goodman, C. A. Micchelli, and J. D. Ward, Spectral radius formulas for subdivision operators, *in* "Recent Advances in Wavelet Analysis" (L. L. Schumaker and G. Webb, Eds.), pp. 335–360, Academic Press, San Diego, 1994.
12. B. Han and R. Q. Jia, Multivariate refinement equations and subdivision schemes, *SIAM J. Math. Anal.* **29** (1998), 1177–1199.
13. C. Heil and D. Colella, Matrix refinement equations: Existence and uniqueness, *J. Fourier Anal. Appl.* **2** (1996), 363–377.
14. C. Heil, G. Strang, and V. Strela, Approximation by translates of refinable functions, *Numer. Math.* **73** (1996), 75–94.
15. R. Q. Jia, Subdivision schemes in L_p spaces, *Adv. Comput. Math.* **3** (1995), 309–341.
16. R. Q. Jia, Characterization of smoothness of multivariate refinable functions in Sobolev spaces, *Trans. Amer. Math. Soc.* **351** (1999), 4089–4112.
17. R. Q. Jia, S. Riemenschneider, and D. X. Zhou, Approximation by multiple refinable functions, *Canad. J. Math.* **49** (1997), 944–962.
18. R. Q. Jia, S. Riemenschneider, and D. X. Zhou, Vector subdivision schemes and multiple wavelets, *Math. Comp.* **67** (1998), 1533–1563.
19. R. Q. Jia, S. Riemenschneider, and D. X. Zhou, Smoothness of multiple refinable functions and multiple wavelets, *SIAM J. Matrix Anal. Appl.* **21** (1999), 1–28.
20. K. S. Lau and M. F. Ma, The regularity of L^p-scaling functions, *Asian J. Math.* **1** (1997), 272–292.
21. K. S. Lau and J. Wang, Characterization of L^p-solution for the two scale dilation equations, *SIAM J. Math. Anal.* **26** (1995), 1018–1046.
22. C. A. Micchelli, Interpolatory subdivision schemes and wavelets, *J. Approx. Theory* **86** (1996), 41–71.
23. C. A. Micchelli and H. Prautzsch, Uniform refinement of curves, *Linear Algebra Appl.* **114–115** (1989), 841–870.
24. C. A. Micchelli and T. Sauer, Regularity of multiwavelets, *Adv. Comput. Math.* **7** (1997), 455–545.
25. G. Plonka, Approximation order provided by refinable function vectors, *Constr. Approx.* **13** (1997), 221–244.
26. G.-C. Rota and G. Strang, A note on the joint spectral radius, *Indag. Math.* **22** (1960), 379–381.
27. G. Strang and D. X. Zhou, Inhomogeneous refinement equations, *J. Fourier Anal. Appl.* **4** (1998), 733–747.
28. L. Villemoes, Wavelet analysis of refinement equations, *SIAM J. Math. Anal.* **25** (1994), 1433–1460.
29. D. X. Zhou, Existence of multiple refinable distributions, *Michigan Math. J.* **44** (1997), 317–329.
30. D. X. Zhou, The p-norm joint spectral radius for even integers, *Methods Appl. Anal.* **5** (1998), 39–54.